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Hemivariational Inequality Modeling of Hybrid Laminates with Unidirectional Composite Constituents

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The overall tensile and bending behavior of unidirectional composite elements with monotone and nonmonotone, possibly multivalued, constitutive laws for each constituent are studied within a nonsmooth mechanics framework. A nondifferentiable and possibly nonconvex, caused by degradation effects, potential energy is formulated for the whole mechanical system. For the structural analysis problem the potential energy minimization problem is considered. The arising variational or hemivariational (in the case of nonmonotone laws) inequality problems are solved by appropriate nonconvex-nonsmooth computational mechanics techniques. Parametric numerical investigations of typical composite elements are presented, which shed light into the complex behavior of the composite structural elements. On the other hand, the arising overall laws can be used as phenomenological laws for structural analysis of large-scale structures incorporating the studied composite elements.

I. Introduction

COMPOSITE materials is a rapidly maturing technology with emerging applications in a wide range of industries far beyond the aerospace domain where they first became popular. Nowadays, composites are popular even in the civil-engineering domain and are used for the construction of light bridges, domes, space trusses, etc. Moreover, composite elements are used in structures that are susceptible to electrochemical actions or corrosion, such as underground facilities, offshore oil platforms, waterways, and harbors.[1]

The basic principles involved in the design of structures made of composite materials are the same as those of isotropic materials such as steel. The classical theories and methods of analysis can be used for the design of composite structures, as far as the constitutive relationship takes into account the material anisotropy and the strength degradation effects, which are present in the majority of composite materials. Moreover, the same basic design knowledge and technique used for other materials as, e.g., for reinforced concrete, can be applied to composite structures. However, the reader should have in mind that the implementation of accurate design methods for steel (which is the material with the most simple constitutive relationship used today in civil-engineering structures) has required a century of research and experience. In this framework it is not only very important to study these new materials at the materials level, but it is also very important to know their behavior at the structural level. (Compare in this respect the combined effects in the study of beam-to-column connections in steel structures, which lead to strong nonlinearities.) Also, the phenomenological (macroscopic) response under certain types of loading is of great importance.

In unidirectional composites the fibers are aligned in one only direction, thus achieving the maximum fiber alignment and the maximum fiber content. As in principle, the strength of a composite structural element increases in proportion to increasing fiber content, this type of composite provides high strength to the direction of the fibers but very low transverse strength. Therefore it is very common to combine unidirectional composite materials in a certain layered arrangement in order to construct structural elements with improved overall behavior, taking advantage of the properties of each of the constituents. These elements are called hybrid laminates.

In this paper we study the overall tensile and bending behavior of such hybrid laminated elements, which have as constituents unidirectional composites. Each unidirectional composite is assumed to have a nonmonotone, possibly multivalued constitutive law, which takes into account the local cracking or crushing effects. More specifically, we consider here hybrid laminates where each layer is made of a material with a different constitutive (stress-strain) relation. Different monotone and nonmonotone stress-strain laws that may include complete vertical branches are assumed to hold for each layer. Because of the just-mentioned local damage phenomena, a highly nonlinear overall mechanical behavior arises. Such structural elements have been studied by analytical or statistical concepts (see, e.g., Refs. 2 and 3).

To use the overall mechanical law for the phenomenological modeling and the computation of large composite structures, a detailed investigation is first done on the structural element scale. This study is presented here for a few model hybrid laminate elements. The structural analysis methods are based on the theory of nonsmooth mechanics.[4,5]

The type of laws appearing in composite materials is a special case of a more general family of mechanical laws, called nonmonotone, multivalued laws (Figs. 1a and 1b). In Refs. 4–8 this type of law gives rise to a new type of variational forms expressed by inequalities, called hemivariational inequalities. Hemivariational inequalities express the principle of virtual or complementary virtual work in inequality form. The theory of hemivariational inequalities leads to the conclusion that local minima of the potential or the complementary energy functional represent equilibrium positions of the structure. However, it is possible that certain solutions of the problem may not be local minima but more general types of points, which make the potential or the complementary energy substationary.[7,9,10]

Although the formulation of a nonmonotone problem as a hemivariational inequality has many advantages concerning its mathematical study that make possible significant progress in this area,[7,11,12] this progress has not yet been matched by similar developments of the numerical approximation methods.[13,14] Indeed, the numerical determination of all local minima of a nonconvex function is still an open problem in numerical optimization.[15] We recall at this point that in the case of monotone laws (Figs. 1c and 1d) the solution of the inequality constrained problem of minimum

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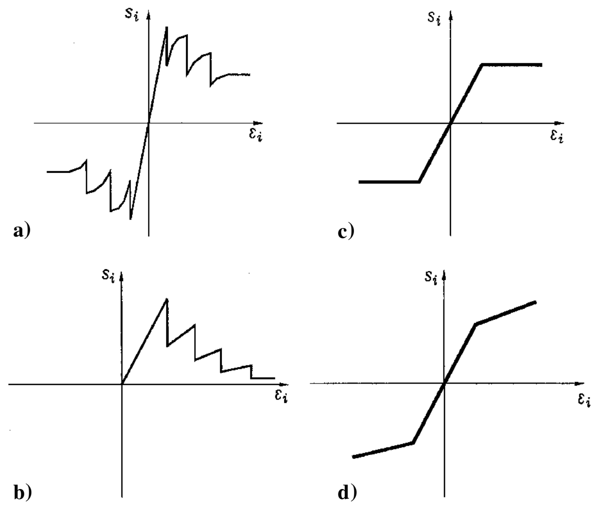


Fig. 1 Various monotone and nonmonotone laws appearing in engineering problems.

potential or complementary energy is obtained easily through a convex optimization algorithm.[4] This point of view is widely accepted within the elastic contact modeling community.[4,10,11,16] This minimum is uniquely determined because of the monotonicity of the stress-strain law.

In this paper two methods are presented for the solution of the hemivariational inequality problem. The first method[10,17–19] approximates the nonconvex problem by a sequence of appropriately defined convex subproblems, which are defined in the course of an iterative algorithm for the determination of the critical points of a nonconvex potential energy function, which can be decomposed into a difference of two convex terms. The second method[10,20–22] is based on the decomposition of the nonconvex problem into a finite number of convex problems that have unique solutions. This approach is based on engineering-inspired approximation schemes and heuristics. Each convex subproblem is numerically treated by convex minimization algorithms such as the ones used for the numerical treatment of classical plasticity problems. In this sense both approaches proposed here can be understood as extensions of the classical plasticity theory, which is familiar to structural engineers.

This treatment sheds light on the complex behavior of the composite structural elements, and on the other hand, the arising overall laws can be used as phenomenological laws for structural analysis of large-scale structures incorporating the studied composite elements. Analogous methods can be used to calculate the elastic region and the degradation effects of composite two- and three-dimensional structural elements. Furthermore, by using appropriate unit cells and homogenization techniques one may extract, by means of numerical experiments, yield surfaces and degradation laws for the studied composites (cf., Ref. 23).

II. Theoretical Formulation

An elastic structure with both classical, linearly elastic and degrading elements is considered. The static analysis problem is described by the following relations.

1) Stress equilibrium equations:

$$\bar{G}\bar{s} = [G \quad G_n]\begin{pmatrix} s \\ s_n \end{pmatrix} = p \tag{1}$$

where $\bar{G}$ is the equilibrium matrix of the discretized structure that takes into account the stress contribution of the linear s and nonlinear s_n elements and p is the loading vector.

2) Strain-displacement compatibility equations:

$$\bar{e} = \bar{G}^T u \qquad \text{or explicitly} \begin{pmatrix} e \\ e_n \end{pmatrix} = \begin{bmatrix} G^T \\ G_n^T \end{bmatrix} u \tag{2}$$

where e and u are the deformation and displacement vectors, respectively.

3) Linear material constitutive law for the structure:

$$s = K_0(e - e_0) \tag{3}$$

where K_0 is the natural and stiffness flexibility matrix and e_0 is the initial deformation vector.

4) Monotone and nonmonotone, superpotential constitutive laws for the nonlinear elements:

$$s_n \in \partial\phi_n(e_n) \tag{4}$$

where $\phi_n(\cdot)$ is a general nonconvex and nondifferentiable potential that produces the law (4) by means of an appropriate generalized differential, set-valued operator ∂ (see, e.g., Ref. 4). Summation over all nonlinear elements gives the total strain energy contribution of them as

$$\Phi_n(e_n) = \sum_{i=1}^{q} \phi_n^{(i)}(e_n) \tag{5}$$

5) Classical support boundary conditions (b.c.):

For the variational formulations of the problem, the virtual work equation is first formulated in a discretized form as

$$s^T(e^* - e) + s_n^T(e_n^* - e_n) = p^T(u^* - u)$$
$$\forall e^*, u^*, e_n^* \text{ s.t. (2), b.c.} \tag{6}$$

Entering the elasticity law (3) into the virtual work equation (6) and using Eq. (2), we get

$$u^T G K_0^T G^T(u^* - u) - (p + GK_0e_0)^T(u^* - u) + s_n^T(e_n^* - e_n) = 0$$
$$\forall u^* \in V_{\text{ad}} = \{v \in \mathbb{R}^n \mid (2), \text{ b.c. hold}\} \tag{7}$$

where $K = G^T K_0^T G$ denotes the stiffness matrix of the structure and $\bar{p} = p + GK_0e_0$ denotes the nodal equivalent loading vector.

At this point we use the weak form of the nonlinear law (4), i.e.,

$$s_n^T(e_n^* - e_n) \le \Phi_n^0(e_n^* - e_n), \qquad \forall e_n^* \tag{8}$$

where $\Phi_n^0(e_n^* - e_n)$ is the directional derivative of the potential Φ_n or in terms of mechanics the virtual work of the nonlinear structural elements for a small deformation equal to $e_n^* - e_n$. Thus, the following hemivariational inequality is obtained:

Find kinematically admissible displacements $u \in V_{\text{ad}}$ such that

$$u^T K(u^* - u) - \bar{p}^T(u^* - u) + \Phi_n^0(u_n^* - u_n) \ge 0, \qquad \forall u^* \in V_{\text{ad}} \tag{9}$$

Equivalently, the potential energy should be stationary at equilibrium; i.e., the structural analysis problem reads as follows:

Find $u \in V_{\text{ad}}$ such that

$$\Pi(u) = \underset{v \in V_{\text{ad}}}{\text{stat}}\left[\Pi(v) = \tfrac{1}{2}v^T K v - \bar{p}^T v + \Phi_n(v)\right] \tag{10}$$

The approach just presented can be specialized to lead to classical, nonlinear minimization problems and variational equalities for differentiable potentials, to variational inequality problems for convex nondifferentiable potentials, and to systems of variational inequalities for difference convex (d.c.) potentials.[4,5,14] Thus all types of nonlinear relations, even with vertical branches (ascending in, e.g., locking effects, and descending in, e.g., degradation ones), can be considered.

Treating the structural analysis problem as a potential energy minimization problem has certain advantages with respect to related limit analysis and optimal design tasks. For instance, an optimal design problem can, in this case, be analyzed within the framework of multilevel optimization. For relevant considerations the reader may see, for instance, Ref. 24 and the references given there. Nevertheless in the last given reference different decompositions of the energy terms are introduced, which have been dictated from the mathematical structure of the problem, whereas in the approach presented here all decompositions are based on mechanical arguments.

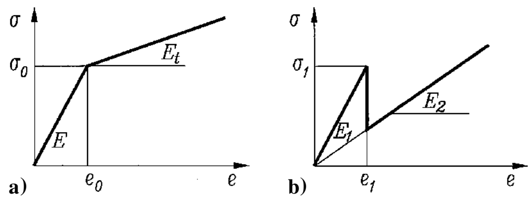


Fig. 2 Stress-strain laws of examples 1 and 2.

Example 1: An elastic linear hardening spring (Fig. 2a) reads as follows:

$$s = Ee \qquad \text{for} \qquad e \leq e_0 \tag{11}$$

$$s = s_0 + E_t(e - e_0) \qquad \text{for} \qquad e \geq e_0 \tag{12}$$

By means of the convex-analysis subdifferential, the spring takes the potential form

$$s \in \partial\phi(e) \tag{13}$$

where the smooth and convex (not strictly) potential $\phi(e)$ is defined as

$$\phi(e) = \begin{cases} \frac{1}{2}Ee^2 & \text{for} \quad e \leq e_0 \\ \frac{1}{2}se - \frac{1}{2}E(e - e_0)^2 + \frac{1}{2}E_t(e - e_0)^2 & \text{for} \quad e \geq e_0 \end{cases}$$

$$= \tfrac{1}{2}Ee^2 + \tfrac{1}{2}(E_t - E)(e - e_0)^2_+ \tag{14}$$

For a softening (degrading) spring, relation (12) holds with $E_t < 0$. The transition from $E_t > 0$ (hardening behavior) to $E_t = 0$ (perfect plastic behavior) and finally to $E_t < 0$ makes an initially strictly convex potential, simply convex, and finally nonconvex. This potential has in every case the form of Eq. (14), which indicates also a way of a possible d.c. decomposition of the corresponding mechanism.

Example 2: A breaking element (Fig. 2b) reads

$$s = E_1 e \qquad \text{for} \qquad e \leq e_1$$

$$s = E_2 e \qquad \text{for} \qquad e \geq e_1 \tag{15}$$

with $E_2 \leq E_1$. This law admits a superpotential writing by means of a nondifferentiable and nonconvex potential, which is nevertheless piecewise differentiable. By using an active index strategy to identify, after examination of the value of e with respect to e_1, which part of the law should be used, the constitutive law can be treated in a classical way. A more compact representation, which is permitted within the nonsmooth mechanics framework, treats relation (15) in a unified way by means of a structured potential. A function that can locally be written as a minimum of two convex constituents or as a difference of two convex functions can be used here:

$$\phi(e) = \min_e[\tilde{\phi}_1(e), \tilde{\phi}_2(e)] \qquad \text{or} \qquad \phi(e) = \phi_1(e) - \phi_2(e) \tag{16}$$

The second form is used in the sequel (see more details in Ref. 14).

III. Algorithmic Approximation of the Nonconvex-Nonsmooth Optimization Problem

In the following a general potential energy minimization problem is considered for the structural analysis problem:

$$\Pi(\boldsymbol{u}) = \min \Pi(\boldsymbol{v}) = \left[\Pi_{\text{in}}(\boldsymbol{e}) + \Phi_1(\boldsymbol{e}) - \Phi_2(\boldsymbol{e}) - \boldsymbol{p}^T\boldsymbol{v}\right] \tag{17}$$

where $\Pi(\boldsymbol{u})$ is the total potential energy of the structure, $\Pi_{\text{in}}(\boldsymbol{e})$ is the internal elastic energy, and $\Phi_1(\boldsymbol{e})$, $\Phi_2(\boldsymbol{e})$ are the convex and concave parts of the potential energy that corresponds to the nonlinear elements. In the small displacement and deformation theory adopted here, the potential remains a difference of convex functions in the displacement variables' space as well:

$$\Pi(\boldsymbol{u}) = \underbrace{\Pi_{\text{in}}(\boldsymbol{u}) + \Phi_1(\boldsymbol{u}) - \boldsymbol{p}^T\boldsymbol{u}}_{\Pi_1} - \underbrace{\Phi_2(\boldsymbol{u})}_{\Pi_2} = \Pi_1(\boldsymbol{u}) - \Pi_2(\boldsymbol{u}) \tag{18}$$

where $\Pi_1(\boldsymbol{u})$ [respectively, $\Pi_2(\boldsymbol{u})$] is the convex (respectively, the concave) constituent of $\Pi(\boldsymbol{u})$.

We recall that the substationarity condition for problem (9) can be resolved by means of the convex subdifferentials of the two convex functions involved in the d.c. potential decomposition, i.e., it reads as follows:

Find $u \in V_{\text{ad}} \subset \mathbb{R}^n$ such that there exists

$$\boldsymbol{w}_2 \in \partial\Pi_2(\boldsymbol{u}) \qquad \text{with} \qquad \boldsymbol{w}_2 \in \partial\Pi_1(\boldsymbol{u}) \tag{19}$$

In other words a stress equilibrium relation must be considered, which takes elements from both potentials introduced by the d.c. decomposition. The structural analysis problem can be treated by the following algorithm:

1) Choose an arbitrary starting point $u_0 \in V_{\text{ad}}$. If condition (19) holds at u_0, then u_0 is an equilibrium point and stop.

2) For $k \geq 0$ take any $w_k \in \partial\Pi_2(u_k)$. Put

$$\Pi_k(u) = \Pi_1(u) - \Pi_2(u_k) - \langle w_k,\ u - u_k\rangle$$

Solve

$$\min_{x \in \mathbb{R}^n}[\Pi_k(u)] = \Pi_k(u_{k+1}) \tag{20}$$

3) If $u_{k+1} = u_k$, stop, else continue with step 2.

The mathematical properties of this algorithm have been studied in Refs. 14 and 25. The mechanical interpretation is obvious: a modified linearly elastic problem is considered with a total potential energy equal to $\Pi_k(u)$ as defined by the contribution of the damaging mechanisms or the concave parts of the initial potential energy, and this modified problem is solved iteratively until convergence. Note that, because of the arbitrariness of a d.c. decomposition of a function, the preceding scheme can be modified to encompass several variants of previously proposed algorithms for the treatment of general hemivariational inequality problems (see, e.g., Refs. 5 and 26).

Another more general approach that can be applied in the case that a d.c. composition of the nonmonotone stress-strain law is not possible is presented in the following. The aim of this approach is to bypass the nonconvex substationarity problem (10) by minimizing a sequence of appropriately defined convex functions in which the nonconvex potential $\Phi_n(\boldsymbol{v})$ has been substituted with the convex potential $p(\boldsymbol{v})$. We consider first the following minimization problems:

$$\min\left[\tfrac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\boldsymbol{v} - \bar{\boldsymbol{p}}^T\boldsymbol{v} + p^{(i)}(\boldsymbol{v})\right] \tag{21}$$

where in each step the convex potential $p^{(i)}(\boldsymbol{v})$ is selected such that the following relation is fulfilled:

$$\partial p^{(i)}\left[\boldsymbol{v}^{(i-1)}\right] = \bar{\partial}\Phi_n\left[\boldsymbol{v}^{(i-1)}\right] \tag{22}$$

If the nonconvex minimization problem is written in the form

$$\min[\Pi(\boldsymbol{v})] = \min\left\{\tfrac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\boldsymbol{v} - \bar{\boldsymbol{p}}^T\boldsymbol{v} + p(\boldsymbol{v}) + [\Phi_n(\boldsymbol{v}) - p(\boldsymbol{v})]\right\} \tag{23}$$

then we can write the following minimization problems:

$$\min\left[\Pi^{(1)}(\boldsymbol{v})\right] = \min\left(\tfrac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\boldsymbol{v} - \bar{\boldsymbol{p}}^T\boldsymbol{v} + p^{(1)}(\boldsymbol{v}) + \left\{\Phi_n\left[\boldsymbol{v}^{(0)}\right] - p^{(1)}\left[\boldsymbol{v}^{(0)}\right]\right\}\right)$$

$$\vdots$$

$$\min\left[\Pi^{(i)}(\boldsymbol{v})\right] = \min\left(\tfrac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\boldsymbol{v} - \bar{\boldsymbol{p}}^T\boldsymbol{v} + p^{(i)}(\boldsymbol{v}) + \left\{\Phi_n\left[\boldsymbol{v}^{(i-1)}\right] - p^{(i)}\left[\boldsymbol{v}^{(i-1)}\right]\right\}\right)$$

$$\vdots$$

$$\min\left[\Pi^{(n)}(\boldsymbol{v})\right] = \min\left(\tfrac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\boldsymbol{v} - \bar{\boldsymbol{p}}^T\boldsymbol{v} + p^{(n)}(\boldsymbol{v}) + \left\{\Phi_n\left[\boldsymbol{v}^{(n-1)}\right] - p^{(n)}\left[\boldsymbol{v}^{(n-1)}\right]\right\}\right) \tag{24}$$

If we suppose that, in the last case we have the convergence of the iterative scheme, then we have that $|\boldsymbol{v}^{(n)} - \boldsymbol{v}^{(n-1)}| \leq \varepsilon$. Because of Eq. (22), the variation of the last term of Eq. (24) with respect to $\boldsymbol{v}$ becomes very small, and finally we can write

$$\text{argmin}[\Pi(\boldsymbol{v})] = \text{argmin}\left[\tfrac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\boldsymbol{v} - \bar{\boldsymbol{p}}^T\boldsymbol{v} + p^{(n)}(\boldsymbol{v})\right] \tag{25}$$

where on the left-hand side we mean the local minimum sought. By means of the preceding relations, it is easily verified that a solution of the initial minimization problem of $\Pi(\boldsymbol{v})$ is obtained using the proposed iterative scheme, but the full proof of convergence remains an open problem. However, in the various numerical experiments we have performed, convergence was always achieved. Therefore, the following heuristic algorithm is formulated:

1) Select a starting point $\boldsymbol{v}^{(0)}$, and initialize i to 1.

2) For the point $\boldsymbol{v}^{(i)}$ select a convex superpotential $p^{(i)}$ such that relation (22) is fulfilled.

3) Find the minimum $\boldsymbol{v}^{(i)}$ of the convex superpotential:

$$\tfrac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\boldsymbol{v} - \bar{\boldsymbol{p}}^T\boldsymbol{v} + p^{(i)}(\boldsymbol{v})$$

4) If $|\boldsymbol{v}^{(i)} - \boldsymbol{v}^{(i-1)}| \leq \varepsilon$, where ε is an appropriate small number, then a substationarity point of Eq. (10) has been determined and terminate the algorithm; else set $i = i + 1$ and repeat step 2.

In an engineering problem where a one-dimensional nonmonotone stress-strain law $h(\boldsymbol{v})$ is involved, the process in step 2 of the preceding algorithm is an easy task because one has to select a monotone law $g(\boldsymbol{v})$ such that $g[\boldsymbol{v}^{(i)}] = h[\boldsymbol{v}^{(i-1)}]$. In general, the convex superpotentials that approximate the nonconvex superpotentials are selected in such a way that the computational effort is minimal. This task depends on the particular nonmonotone potentials to be approximated. Notice also that different starting points for the preceding algorithm can lead to different substationarity points of the nonconvex superpotential.

More general types of algorithms and the relation with other nonsmooth computational mechanics methods are discussed in Ref. 10.

IV. Numerical Applications

A. Numerical Implementation

To solve the examples presented in the next sections, a computer program was written in FORTRAN in which the approach already presented was implemented within the finite element framework. All of the examples presented next involve frame elements, and therefore the numerical scheme involved a standard two-dimensional beam element with the well-known matrices found in classical structural analysis textbooks. The nonlinearity was restricted to the σ-ε relationship and was handled numerically by restricting the N-u relation. To approximate the nonconvex superpotentials introduced by the nonmonotone laws, convex superpotentials were used and, more specifically, the convex potentials that correspond to elastic-plastic type laws. Therefore, the nonmonotone law is approximated by standard elastic-plastic monotone laws. As described in the preceding section, the central problem of the proposed numerical scheme is reduced from a nonconvex optimization problem to the solution of standard elastoplasticity problems. Regarding the solution of the elastoplasticity problem, a Lagrange multiplier formulation was used. The constitutive equations read as follows:

$$\boldsymbol{e}_E = \boldsymbol{F}_0\boldsymbol{s} \tag{26}$$

$$\boldsymbol{e} = \boldsymbol{e}_E + \boldsymbol{e}_p \tag{27}$$

$$\boldsymbol{e}_p = \boldsymbol{N}^T\lambda \tag{28}$$

$$\Phi = \boldsymbol{N}\boldsymbol{s} - \boldsymbol{k} \tag{29}$$

$$\Phi^T\lambda = 0 \tag{30}$$

$$\Phi \leq 0 \tag{31}$$

$$\lambda \geq 0 \tag{32}$$

where $\boldsymbol{F}_0$ represents the natural flexibility matrix of the structure. Moreover, $\boldsymbol{e}$ represents the strain vector consisting of two parts, the elastic strain $\boldsymbol{e}_E$ and the plastic strain $\boldsymbol{e}_p$. Furthermore, λ is the plastic multiplier vector, Φ represents the yield functions, $\boldsymbol{N}$ is the matrix of the gradients of the yield functions with respect to the stresses, and $\boldsymbol{k}$ is a vector consisting of positive constants that correspond in this specific case to the limit of the axial force. In the case treated here, matrix $\boldsymbol{N}$ is a simple matrix selecting from the stress vector $\boldsymbol{s}$ the nonlinear constituents (in our case the axial force $\boldsymbol{N}$ of the elements).

The preceding equations together with the equilibrium equations and the compatibility relations yield the following minimization problem:

$$\min\Big(\Pi = \tfrac{1}{2}\boldsymbol{u}^T\boldsymbol{K}\boldsymbol{u} - \boldsymbol{p}^T\boldsymbol{u} - \tfrac{1}{2}\lambda^T\boldsymbol{N}\boldsymbol{K}_0\boldsymbol{N}^T\lambda - \boldsymbol{u}^T\boldsymbol{G}\boldsymbol{K}_0\boldsymbol{N}^T\lambda + \lambda^T\boldsymbol{k},\ \lambda \geq 0\Big) \tag{33}$$

In Eq. (33) $\boldsymbol{G}$ is the equilibrium matrix of the structure, $\boldsymbol{K}_0$ is the natural stifness matrix, $\boldsymbol{p}$ is the loading vector, and $\boldsymbol{u}$ is the displacement vector. In the preceding problem Π is the potential energy function of the structure, which can also be written in the form

$$\Pi = \tfrac{1}{2}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} - \boldsymbol{c}^T\boldsymbol{x} \tag{34}$$

where

$$\boldsymbol{x} = \begin{pmatrix} \boldsymbol{u} \\ -\lambda \end{pmatrix} \tag{35}$$

$$\boldsymbol{c} = \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{k} \end{pmatrix} \tag{36}$$

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{G} \\ \boldsymbol{N} \end{pmatrix}\boldsymbol{K}_0[\boldsymbol{G}^T \quad \boldsymbol{N}^T] \tag{37}$$

The solution of the preceding minimization problem can be achieved by using a standard quadratic programming procedure that can be found in well-known mathematical routine libraries (e.g. IMSL, HARWELL, NAG). In the FORTRAN code that we implemented, the quadratic optimization routine VE09AD of the HARWELL subroutine library was used.

B. Unidirectional Composites Under Axial Loading

Here we consider one-dimensional, hybrid-layered composites where each layer is made of a unidirectional composite or steel. The following laminates are examined: 1) hybrid laminate in tension made of a layer of aramid fiber composite (AFC) backed up with steel plates and 2) hybrid laminate in tension made of a layer of AFC backed up with carbon fiber composite (CFC) plates.

The elasticity modulus for the AFC is equal to $E_{A,L} = 71{,}000$ N/mm^2, and the breaking point lies at $\varepsilon_{B,\mathrm{AFC}} = 2\%$. For the steel plates we have $E_{\mathrm{st},L} = 125{,}000$ N/mm^2 and yield limit $\sigma_{\mathrm{st},y} = 250$ N/mm^2. For the CFC we consider $E_{C,L} = 133{,}000$ N/mm^2 and breaking point at $\epsilon_{B,\mathrm{CFC}} = 1.2\%$. The stress-strain laws for each material are depicted in Fig. 3.

The arrangement and dimensions of the specimen are depicted in Fig. 4a. To analyze the preceding problem, the model of Fig. 4b was used. The model consists of two elements, one representing the AFC core and the other representing the external CFC or steel plates. For each problem several different cases are considered by changing the thickness of each layer. The characteristic parameter is the ratio c/t, where c is the thickness of the AFC core and t is the total thickness of the three layers. For each case the cross-sectional area of each element is accordingly assigned.

The results obtained by the application of the methods already outlined to the preceding problem are depicted in Figs. 5 and 6. Figure 5 gives the one-dimensional overall σ-ε response of the steel-AFC combination. We notice that the damage initiates with the plastification of the steel plates, and then, by constant loading, the structural element is deformed up to the breaking of the AFC core.

The behavior of the CFC-AFC-hybrid composite is given in Fig. 5. In this case primary damage arises through the breaking of the brittle carbon fibers. Additional increasing of the loading until the breaking of the aramide fibers is possible only for composite elements with $t_{\mathrm{AFC}}/t > 75\%$. If the structural element is used in a

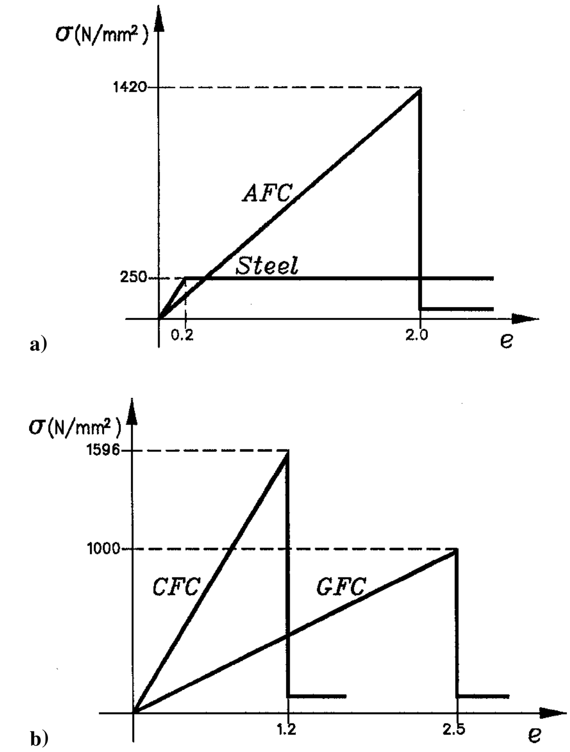


Fig. 3 Stress-strain laws for the AFC, steel, CFC, and GFC.

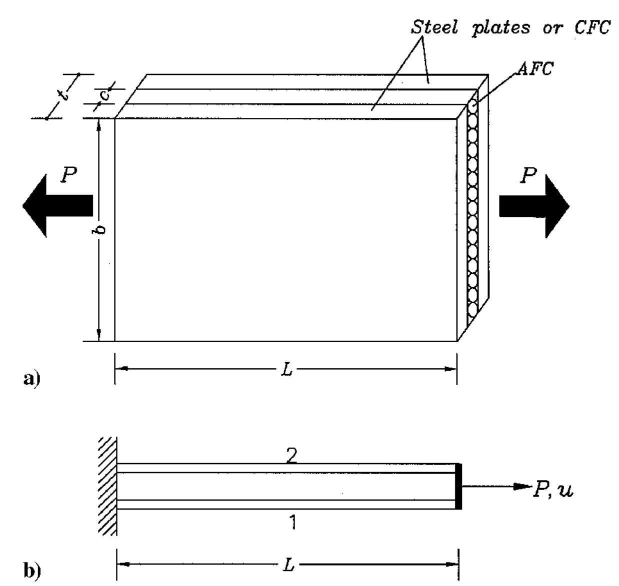


Fig. 4 Unidirectional composite under axial loading and the corresponding analysis model.

statically indeterminate structure, then a reduced remaining strength can be used (after the first carbon-fiber damage).

C. Unidirectional Composites in Bending

Here we consider a layered, hybrid, structural element in bending made of CFC and glass fiber composite (GFC). The core is made of GFC and the external layers of CFC.

The material constants for the GFC are $E_{G,L}$ =40,000 N/mm^2 and $\varepsilon_{B,C}$ =2.5% (see Fig. 3b).

The arrangement and dimensions of the specimen are given in Fig. 7a. To analyze the problem, the model of Fig. 7b was used. The model consists of 20 parallel beam elements, which are connected with a rigid bar. This model was selected to enforce the Bernoulli beam conditions (deformed cross section remains plane). Each element has a constant cross-sectional area equal to $bt/20$. To analyze

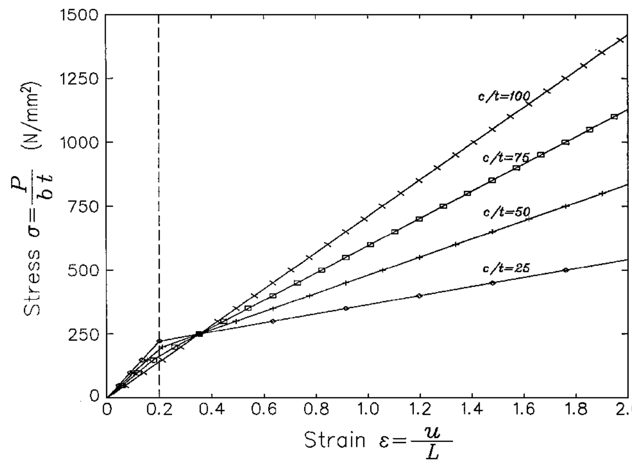


Fig. 5 Stress-strain curves for the steel-AFC combination.

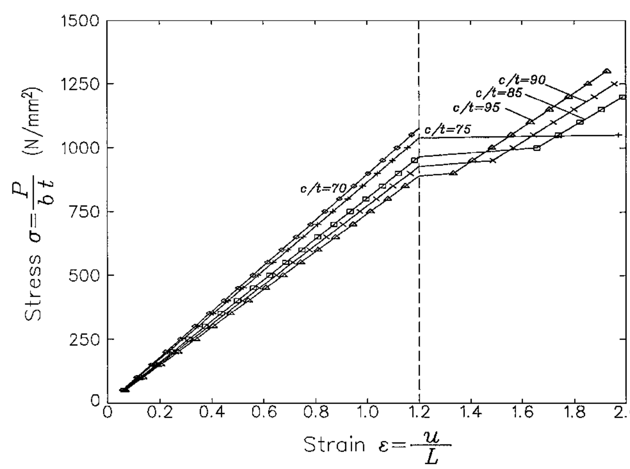


Fig. 6 Stress-strain curves for the CFC-AFC composite.

different cases of c/t (core thickness to total thickness), we assign different material properties to each element. For example, to consider the case CFC-GFC-CFG, c/t =0.70, we assign the material properties of the CFC to elements 1, 2, 3, 18, 19, 20 and the material properties of the GFC to the rest of the elements.

Each one of the preceding problems was analyzed using the algorithms outlined in the preceding section. Because of the particular characteristics of the problem (multiple minima for the same load level), different starting points had to be defined to obtain the points beyond the breaking of the CFC (ε =1.2%) for all of the cases of c/t except the one that corresponds to c/t =1.00.

The equivalent stress ($6M/t^2$)-extreme strain diagrams for the CFC-GFC-CFC combination are given in Fig. 8. The brittle CFC, placed at the external layers, is the first constituent to break and to delaminate. A further increasing of the loading is possible only by relatively large core thickness (proportion of the glass laminate, i.e., $h/t > 0.9$).

It is very interesting for this problem to examine the potential energy functions. For this reason we considered the cases c/t =0.60 and 0.80. Figure 9a gives the potential energy curves for the structure with c/t =0.60 and for several values of the external loading. For low values of the loading σ =100, the solution is unique, as is also verified from Fig. 7. For higher values of the loading (σ =150 − 350), the substationarity problem has two solutions, one in the interval $0 \le \varepsilon \le 1.2$ and one for $\varepsilon > 1.2$. For even higher values of the loading, the problem has only one solution that corresponds to the structure with both materials assumed as linear elastic. The same results hold for the case c/t =0.80 (Fig. 9b). In this case, because of the greater proportion of the GFC (which has a higher ultimate strain than the brittle CFC), the undertaken loading is significantly higher.

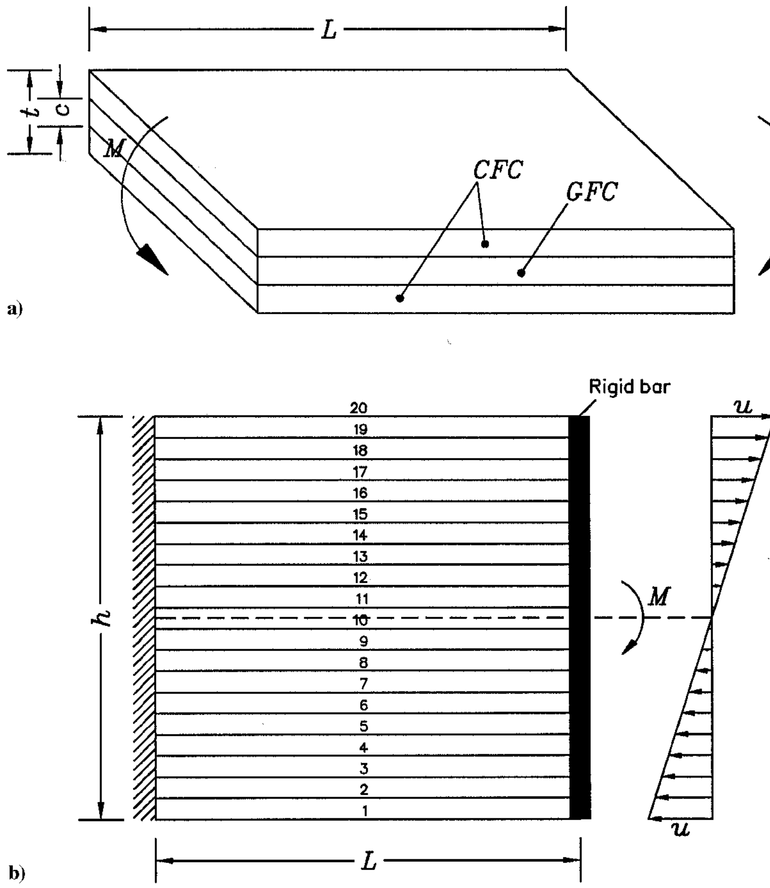


Fig. 7 Unidirectional composite in bending and the corresponding analysis model.

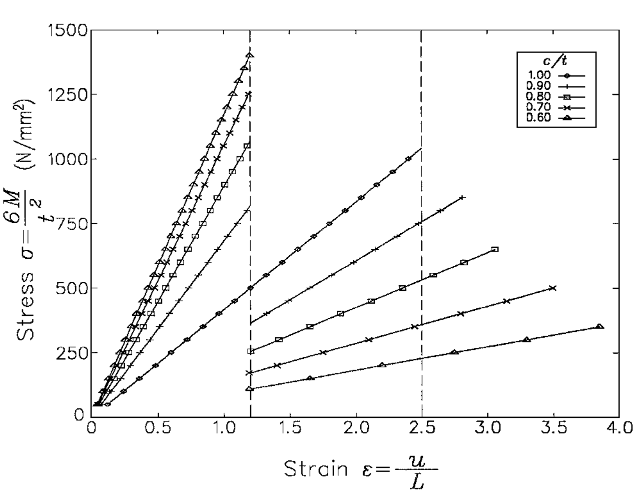
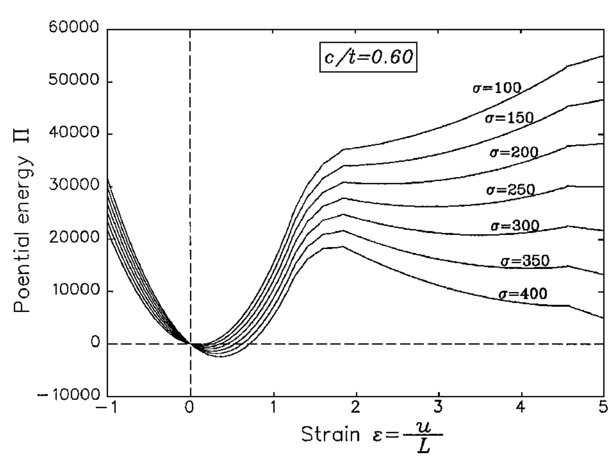


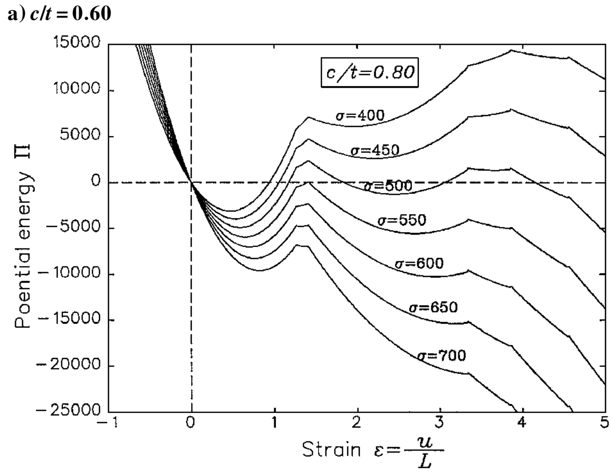
Fig. 8 Stress-strain curves for the GFC-CFC composite.

Concerning the efficiency of the proposed numerical schemes, both algorithms converged quickly to the solution of the problem. All of the presented problems were solved within two to three steps of the iterative algorithms presented in the preceding section with a relative accuracy of 10^{-4} with respect to a second-order norm of the displacements, i.e., the algorithms are assumed to converge when $\|v^{(i)} - v^{(i-1)}\|/\|v^{(i-1)}\| \leq 10^{-4}$.

The parametric investigation of the composite elements, which has been used as a numerical example here, reproduces the semianalytical and numerical results of Ref. 2. In this reference the results have also been compared with experimental measurements. The advantage of the approach presented here is that all phase transitions connected with damaging and partial degradation effects are automatically captured by the mechanical model and the corresponding algorithm.



a) *c/t* = 0.60



b) *c/t* = 0.80

Fig. 9 Potential energy curves for two GFC-CFC composites.

V. Conclusions

A hemivariational inequality modeling of the complete nonlinear static mechanical behavior of composites is outlined. Appropriate algorithms are given for the numerical solution of the arising problem within the framework of a finite element discretization. Model specimens of hybrid laminates with unidirectional composite constituents are analyzed by means of the outlined structural analysis techniques. The studied problems exhibit highly nonlinear mechanical behavior, with falling branches in the overall load-displacement or moment-rotation laws, that is caused by strength degradation of the composite. The numerical treatment of these effects requires specialized algorithms, and the ones outlined and demonstrated here are appropriate for this task. Their use in homogenization problems, limit analysis problems, failure surfaces' (or criteria) determination, or optimal design tasks for a given composite configuration should be possible for interested readers.

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